Review Study of Dynamic Relaxation Numerical Technique Used in Bending of Laminated Composite Plates

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Abstract

The method of dynamic relaxation (DRM) in its early stages of development was perceived as a numerical finite difference technique. It was first used to analyse structures, then skeletal and cable structures, and plates. The method relies on a discretized continuum in which the mass of the structure is assumed to be concentrated at given points (i.e. nodes) on the surface. The system of concentrated masses oscillates about the equilibrium position under the influence of out of balance forces. With time, it comes to rest under the influence of damping. The iterative scheme reflects a process, in which static equilibrium of the system is achieved by simulating a pseudo dynamic process in time. In its original form, the method makes use of inertia term, damping term and time increment.

Keywords: DRM, bending, composite laminates, plate equations, finite differences

Introduction

In the present work, finite differences coupled with dynamic relaxation (DRM) Method, which is a numerical technique, is used. The DRM method was first proposed and developed in 1960; see Rushton KR [1], Cassell AC & Hobbs RE [2], Day AS [3]. In this method, the equations of equilibrium are converted to dynamic equations by adding damping and inertia terms. These are then expressed in finite difference form and the solution is obtained by an iterative procedure as explained below.

Dynamic Relaxation (DRM) Formulation

The DRM formulae begin with the dynamic equation, which can be written as:

$$f = \rho \frac{\partial^2 u}{\partial t^2} + k \frac{\partial u}{\partial t} \qquad (1)$$

Where f is a function of the stress resultants and couples, u is referred to as a displacement, $\partial u / \partial t$ and $\partial^2 u / \partial t^2$ are the velocity and acceleration respectively, and k are the inertia and damping coefficients respectively, and t is time.

If the velocities before and after a period Δt at an arbitrary node in the finite difference mesh are denoted by $\left\{\frac{\partial u}{\partial t}\right\}_{n-1}$ and $\left\{\frac{\partial u}{\partial t}\right\}_{n}$ respectively, then using finite differences in time, and specifying the value of the function at $\left(n - \frac{1}{2}\right)$, it is possible to write equation (1) in the following form:

$$f_{n-\frac{1}{2}} = \frac{\rho}{\Delta t}\left[\left\{\frac{\partial u}{\partial t}\right\}_{n} - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] + k\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}} \qquad (2)$$

The term $\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}}$ represents the velocity at the middle of the time increment, which can be approximated by the mean of the velocities before and after the time increment Δt, i.e.

$$\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}} = \frac{1}{2}\left[\left\{\frac{\partial u}{\partial t}\right\}_{n} + \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right]$$

Hence, equation (2) can be expressed as:

$$f_{n-\frac{1}{2}} = \frac{\rho}{\Delta t}\left[\left\{\frac{\partial u}{\partial t}\right\}_n - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] + \frac{k}{2}\left[\left\{\frac{\partial u}{\partial t}\right\}_n + \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] \quad (3)$$

Equation (3) can then be arranged to give the velocity after the time interval, Δt :

$$\left\{\frac{\partial u}{\partial t}\right\}_n = (1+k^*)^{-1}\left[\frac{\Delta t}{\rho}f_{n-\frac{1}{2}} + (1-k^*)\left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] \quad (4)$$

Where $\quad k^* = \frac{k\Delta t}{2\rho}$

The displacement at the middle of the next time increment can be determined by integrating the velocity, so that:

$$u_{n+\frac{1}{2}} = u_{n-\frac{1}{2}} + \left\{\frac{\partial u}{\partial t}\right\}_n \Delta t \quad (5)$$

The formulation is completed by computing the stress resultants and stress couples from the known displacement field. The iterative procedure begins at time t=0 with all initial values of the velocities, displacements, and stresses equal to zero or any suitable values. In the first iteration, the velocities are obtained from equation (4), and the displacements from equation (5). After the satisfaction of the displacement boundary conditions, the stress resultants and stress couples are computed and the appropriate boundary conditions for stresses are applied. Subsequent iterations follow the same steps. The iterations continue until the desired accuracy is achieved.

The Plate Equations

Dimensional plate equations

The equations concerning the analysis of plates in bending (i.e. Equations (6), (7), (8), (9), and (10) are derived in Turvey & Osman [4-6] and Osama Khayal [7-9]. The first integrated equations of equilibrium can then be written in the following form:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = 0 \quad (6)$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = 0 \quad (7)$$

Hence the integrated moment equilibrium equations are:

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 = 0 \quad (8)$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 = 0 \quad (9)$$

$$N_1\frac{\partial^2 w}{\partial x^2} + 2N_6\frac{\partial^2 w}{\partial x\partial y} + N_2\frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} + q = 0 \quad (10)$$

Non-dimensional plate equations

The plate equations can be written in non-dimensional form as follows:

$$
\begin{aligned}
&[x', y', \theta'] = \tfrac{1}{a}[x, y, b], \quad h' = \tfrac{h}{b} = 1, \; w' = \tfrac{w}{h}\\
&[u', v'] = \left(\tfrac{a}{h^2}\right)[u, v], \; [\phi', \psi'] = \left(\tfrac{a}{h}\right)[\phi, \psi]\\
&c_i' = \left(\tfrac{a^2}{h}\right)c_i, \; z_i' = \left(\tfrac{a^2}{h}\right)z_i, \; i = 1,2,6\\
&A_i' = \left(\tfrac{1}{E_2 h}\right)A_i, \; B_i' = \left(\tfrac{1}{E_2 h^2}\right)B_i, \\
&D_i' = \left(\tfrac{1}{E_2 h^3}\right)D_i, \; (j = 1,2,6)\\
&N_i' = \left(\tfrac{a^2}{E_2 h^3}\right)N_i, \; M_i' = \left(\tfrac{a^2}{E_2 h^3}\right)M_i, \; i = 1,2,6\\
&Q_i' = \left(\tfrac{a^3}{E_2 h^3}\right)Q_i, \; (i = 1,2), \; q' = \left(\tfrac{a^4}{E_2 h^3}\right)q
\end{aligned}
\quad (3.6)
$$
$$(11)$$

Substituting equation (11) into equations (6), (7), (8), (9) and (10) then the non-dimensional dynamic plate equations are obtained. In the following equations, the primes are omitted:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = \rho_u\frac{\partial^2 u}{\partial t^2} + k_u\frac{\partial u}{\partial t} \quad (3.7)$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = \rho_v\frac{\partial^2 v}{\partial t^2} + k_v\frac{\partial v}{\partial t} \quad (3.8)$$

$$N_1\frac{\partial^2 w}{\partial x^2} + 2N_6\frac{\partial^2 w}{\partial x\partial y} + N_2\frac{\partial^2 w}{\partial y^2} + \left(\frac{a}{h}\right)^2\left[\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}\right] + q = \rho_w\frac{\partial^2 w}{\partial t^2} + k_w\frac{\partial w}{\partial t} \quad (3.9)$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - \left(\frac{a}{h}\right)^2 Q_1 = \rho_\phi\frac{\partial^2 \phi}{\partial t^2} + k_\phi\frac{\partial \phi}{\partial t} \quad (3.10)$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - \left(\frac{a}{h}\right)^2 Q_2 = \rho_\psi\frac{\partial^2 \psi}{\partial t^2} + k_\psi\frac{\partial \psi}{\partial t} \quad (3.11)$$

$$(12)$$

The next step is to transform the differential equations into finite difference equations.

The Finite Difference Approximation

Interpolating function F (x, y)

It can be shown by using Taylor's series that the first and second derivatives of a function F(x, y) at an arbitrary node i, j shown in figure 1 can be written as follows:

$$\frac{\partial F}{\partial x}(i, j) = \frac{1}{2\Delta x}\left[F(i+1, j) - F(i-1, j)\right] \quad (13)$$
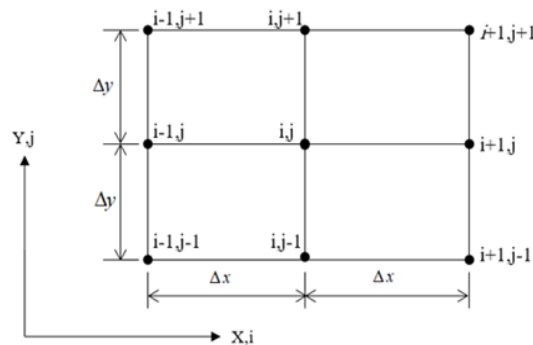

Figure 1: Finite Difference Mesh for an Interpolating Function F(x, y) with Two Independent Variables x, y.

$$\frac{\partial^2 F}{\partial x^2}(i,j) = \frac{1}{\Delta x^2}\left[F(i+1,j) - 2F(i,j) + F(i-1,j)\right] \qquad (14)$$

$$\frac{\partial^2 F}{\partial x \partial y}(i,j) = \frac{1}{4\Delta x \Delta y}\left[F(i+1,j+1) + F(i+1,j-1) - F(i-1,j+1) + F(i-1,j-1)\right] \qquad (15)$$

Also, $\partial F(i,j)/\partial y$ and $\partial^2 F(i,j)/\partial y^2$ can be obtained similarly.

Finite Difference Form of Plate Equations

The velocity equations

According to equation (4) and from the equations of motion of the plate, i.e. equations (12), the velocities are determined as follows:-

$$\frac{du}{dt}(i,j)_n = (1+k_u^*)^{-1}\left[(1-k_u^*)\frac{du}{dt}(i,j)_{n-1} + \frac{\Delta t}{\rho_u(i,j)}F_u(i,j)\right] \qquad (16)$$

$$\frac{dv}{dt}(i,j)_n = (1+k_v^*)^{-1}\left[(1-k_v^*)\frac{dv}{dt}(i,j)_{n-1} + \frac{\Delta t}{\rho_v(i,j)}F_v(i,j)\right] \qquad (17)$$

$$\frac{dw}{dt}(i,j)_n = (1+k_w^*)^{-1}\left[(1-k_w^*)\frac{dw}{dt}(i,j)_{n-1} + \frac{\Delta t}{\rho_w(i,j)}F_w(i,j)\right] \qquad (18)$$

$$\frac{d\phi}{dt}(i,j)_n = (1+k_\phi^*)^{-1}\left[(1-k_\phi^*)\frac{d\phi}{dt}(i,j)_{n-1} + \frac{\Delta t}{\rho_\phi(i,j)}F_\phi(i,j)\right] \qquad (19)$$

$$\frac{d\psi}{dt}(i,j)_n = (1+k_\psi^*)^{-1}\left[(1-k_\psi^*)\frac{d\psi}{dt}(i,j)_{n-1} + \frac{\Delta t}{\rho_\psi(i,j)}F_\psi(i,j)\right] \qquad (20)$$

Where $k_f^* = \frac{k_f \Delta t}{2\rho_f(i,j)}$ and f is denoted by u, v, w, $\frac{dv}{dt}$ or ψ.

In equations (16) _ (20), F1(i,j) , F2(i,j), F3(i,j), F4(i,j) and F5(i,j) are the finite difference approximations of the terms on the left hand side of the dynamic equations (12), i.e.

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = \rho_u \frac{\partial^2 u}{\partial t^2} + k_u \frac{\partial u}{\partial t} \qquad (3.7)$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = \rho_v \frac{\partial^2 v}{\partial t^2} + k_v \frac{\partial v}{\partial t} \qquad (3.8)$$

$$N_1 \frac{\partial^2 w}{\partial x^2} + 2N_6 \frac{\partial^2 w}{\partial x \partial y} + N_2 \frac{\partial^2 w}{\partial y^2} + \left(\frac{a}{h}\right)^2\left[\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}\right] + q = \rho_w \frac{\partial^2 w}{\partial t^2} + k_w \frac{\partial w}{\partial t} \qquad (3.9)$$

(21)

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - \left(\frac{a}{h}\right)^2 Q_1 = \rho_\phi \frac{\partial^2 \phi}{\partial t^2} + k_\phi \frac{\partial \phi}{\partial t} \qquad (3.10)$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - \left(\frac{a}{h}\right)^2 Q_2 = \rho_\psi \frac{\partial^2 \psi}{\partial t^2} + k_\psi \frac{\partial \psi}{\partial t} \qquad (3.11)$$

The displacement equations

The displacements are obtained using the velocities that are explained in Eqns. (16) to (20) as follows:

$$f(i,j)_{n+\frac{1}{2}} = f(i,j)_{n-\frac{1}{2}} + \frac{d\alpha}{dt}(i,j)_n \Delta t \qquad (22)$$

Where can be denoted by u, v, w, $\frac{dv}{dt}$ or ψ.

The stress resultants and couples equations

The finite difference approximations of the stress resultants and stress couples can be obtained using the following two equations (23) and (24) as stated below.

$$\left\{\begin{matrix} N_i \\ M_i \end{matrix}\right\} = \left[\begin{matrix} A_j & B_j \\ B_j & D_j \end{matrix}\right]\left\{\begin{matrix} \varepsilon_j^o \\ \chi_j^o \end{matrix}\right\} \qquad (23)$$

$$\left\{\begin{matrix} Q_2 \\ Q_1 \end{matrix}\right\} = \left[\begin{matrix} A_4 & A_5 \\ A_5 & A_5 \end{matrix}\right]\left\{\begin{matrix} \varepsilon_4 \\ \varepsilon_5 \end{matrix}\right\} \qquad (24)$$

Hence, the finite difference approximations of the stress resultants and stress couples equations can be represented in the form:

$$N_1(i,j) = A_1 \varepsilon_1^o(i,j) + A_2 \varepsilon_2^o(i,j) + A_6 \varepsilon_6^o(i,j) + B_1 \chi_1^o(i,j) + B_2 \chi_2^o(i,j) + B_6 \chi_6^o(i,j) \qquad (25)$$

$$N_2(i,j) = A_2 \varepsilon_1^o(i,j) + A_2 \varepsilon_2^o(i,j) + A_6 \varepsilon_6^o(i,j) + B_2 \chi_1^o(i,j) + B_2 \chi_2^o(i,j) + B_6 \chi_6^o(i,j) \qquad (26)$$

$$N_6(i,j) = A_6 \varepsilon_1^o(i,j) + A_6 \varepsilon_2^o(i,j) + A_6 \varepsilon_6^o(i,j) + B_6 \chi_1^o(i,j) + B_6 \chi_2^o(i,j) + B_6 \chi_6^o(i,j) \qquad (27)$$

$$M_1(i,j) = B_1 \varepsilon_1^o(i,j) + B_2 \varepsilon_2^o(i,j) + B_6 \varepsilon_6^o(i,j) + D_1 \chi_1^o(i,j) + D_2 \chi_2^o(i,j) + D_6 \chi_6^o(i,j) \qquad (28)$$

$$M_2(i,j) = B_2 \varepsilon_1^o(i,j) + B_2 \varepsilon_2^o(i,j) + B_6 \varepsilon_6^o(i,j) + D_2 \chi_1^o(i,j) + D_2 \chi_2^o(i,j) + D_6 \chi_6^o(i,j) \qquad (29)$$

$$M_6(i,j) = B_6 \varepsilon_1^o(i,j) + B_6 \varepsilon_2^o(i,j) + B_6 \varepsilon_6^o(i,j) + D_6 \chi_1^o(i,j) + D_6 \chi_2^o(i,j) + D_6 \chi_6^o(i,j) \qquad (30)$$

$$Q_1(i,j) = A_5 \varepsilon_4^o(i,j) + A_5 \varepsilon_5^o(i,j) \qquad (31)$$

$$Q_2(i,j) = A_4 \varepsilon_4^o(i,j) + A_5 \varepsilon_5^o(i,j) \qquad (32)$$

Estimation of the fictitious variables

To compute the derivatives of displacements, stress resultants and stress couples at the boundaries of a plate, fictitious nodes are considered by extending the finite difference mesh beyond the boundaries as shown in Figure 2. The values of the variables at the fictitious nodes are known as fictitious values. The fictitious values are estimated in order to eliminate the third derivative of the interpolating function, which is quadratic.

Referring to Figure 2, the fictitious values at the points defined by (1, j) can be obtained by extrapolation as follows:
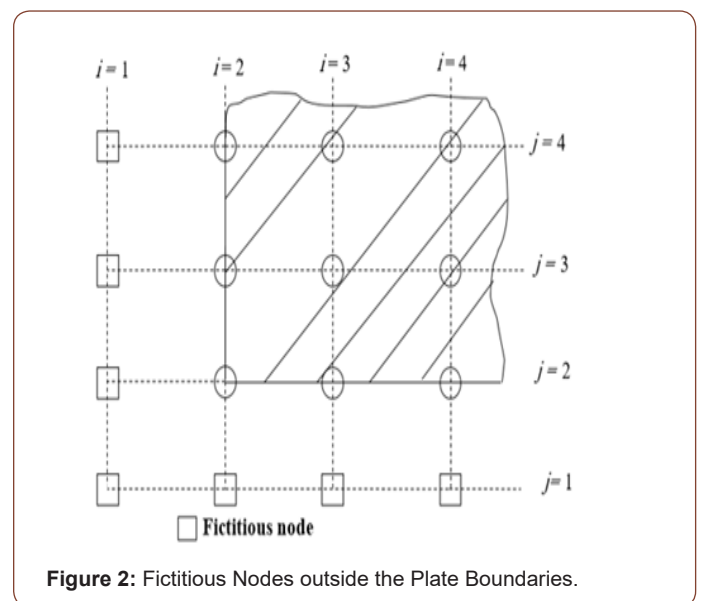

Figure 2: Fictitious Nodes outside the Plate Boundaries.

$$f(1, j) = 3 f(2, j) - 3 f(3, j) + f(4, j) \qquad (33)$$

Where f in equation (33) can be replaced by u, v, w, Ø and Ψ.

The DRM Iterative Procedure

In the DRM technique, explained in the previous sections of this chapter, the static equations of the plate have been converted to dynamic equations i.e. equations (12). Then the inertia and damping terms are added to all of these equations. The iterations of the DRM technique can then be carried out in following procedures:

1. Set all initial values of variables to zero.

2. Compute the velocities from equations (16)-(20).

3. Compute the displacements from equations (22).

4. Apply suitable boundary conditions for the displacements.

5. Compute the stress resultants and stress couples from equations (25) to (32).

6. Apply the appropriate boundary conditions for the stress resultants and stress couples.

7. Check if the convergence criterion is satisfied, if it is not repeat the steps from 2 to 6.

It is obvious that this method requires five fictitious densities and a similar number of damping coefficients so as the solution will be converged correctly.

The Fictitious Densities

The computation of the fictitious densities based on the Gershgorin upper bound of the stiffness matrix of a plate is discussed in Cassel & Hobbs [2]. The fictitious densities vary from point to point over the plate as well as for each iteration, so as to improve the convergence of the numerical computations. The corresponding expressions for the computations of the fictitious densities are given below:

The following fictitious densities have been derived using the procedure proposed by Cassel & Hobbs [2]. With reference to Figure 2.

$$\rho_u(i,j) = 0.5 \left\{ \frac{1}{2\Delta x}\left[\overline{N}_1(i+1,j) + \overline{N}_1(i-1,j)\right] + \frac{1}{2\Delta y}\left[\overline{N}_6(i,j+1) + \overline{N}_6(i,j-1)\right] \right\} \qquad (34)$$

$$\rho_v(i,j) = 0.5 \left\{ \frac{1}{2\Delta x}\left[\overline{N}_6(i+1,j) + \overline{N}_6(i-1,j)\right] + \frac{1}{2\Delta y}\left[\overline{N}_2(i,j+1) + \overline{N}_2(i,j-1)\right] \right\} \qquad (35)$$

$$\rho_w(i,j) = 0.5 \left\{ \left(\frac{a}{h}\right)^2 \left(\frac{\overline{Q}_1}{\Delta x} + \frac{\overline{Q}_2}{\Delta y}\right) + \frac{4}{\Delta y^2}|N_2(i,j)| + \frac{4}{\Delta x^2}|N_1(i,j)| \right.$$
$$+ \frac{2}{\Delta x \Delta y}|N_6(i,j)| + \frac{\overline{N}_1(i,j)}{\Delta x^2}|w(i+1,j) - 2w(i,j) + w(i-1,j)|$$
$$+ \frac{\overline{N}_2(i,j)}{\Delta y^2}|w(i,j+1) - 2w(i,j) + w(i,j-1)|$$
$$\left. + \frac{\overline{N}_6(i,j)}{2\Delta x \Delta y}|w(i+1,j+1) - w(i-1,j+1) + w(i-1,j-1) - w(i+1,j-1)| \right\} \qquad (D.3)$$

$$\rho_\emptyset(i,j) = 0.5 \left\{ \frac{1}{2\Delta x}\left[\overline{M}_1(i+1,j) + \overline{M}_1(i-1,j)\right] \right.$$
$$\left. + \frac{1}{2\Delta y}\left[\overline{M}_6(i,j+1) + \overline{M}_6(i,j-1)\right] + \left(\frac{a}{h}\right)^2 \overline{Q}_1(i,j) \right\} \qquad (D.4) \quad (37)$$

$$+ \frac{3\nabla\lambda}{1}\left[\underline{w}^5(1^\backprime,1) + \underline{w}^5(1^\backprime,-1)\right] + \left(\frac{\psi}{\sigma}\right)\overline{Q}^5(1^\backprime,1) \right\} \qquad (D.2)$$
$$\qquad (38)$$
$$\flat^{th}(1^\backprime,1) = 0.5 \left\{ \frac{3\nabla\lambda}{1}\left[\underline{w}^\varrho(1+1^\backprime,1) + \underline{w}^\varrho(1-1^\backprime,1)\right] \right\}$$

Where the over lined quantities are given by:

$$\overline{N}_1(i,j) = \frac{1}{\Delta x}(A_1 + A_1) + \frac{1}{\Delta y}(A_2 + A_2) + \frac{4}{\Delta x^2}B_1 + \frac{4}{\Delta y^2}B_2 + \frac{2}{\Delta x \Delta y}B_6$$
$$+ A_1 I_1 + A_2 I_2 + A_3 I_3$$

$$\overline{N}_2(i,j) = \frac{1}{\Delta x}(A_2 + A_3) + \frac{1}{\Delta y}(A_2 + A_3) + \frac{4}{\Delta x^2}B_2 + \frac{4}{\Delta y^2}B_2 + \frac{2}{\Delta x \Delta y}B_6$$
$$+ A_2 I_1 + A_2 I_2 + A_3 I_3$$

$$\overline{N}_6(i,j) = \frac{1}{\Delta x}(A_3 + A_3) + \frac{1}{\Delta y}(A_3 + A_3) + \frac{4}{\Delta x^2}B_6 + \frac{4}{\Delta y^2}B_6 + \frac{2}{\Delta x \Delta y}B_6$$
$$+ A_3 I_1 + A_3 I_2 + A_3 I_3$$
$$+ B^3 I^1 + B^3 I^3 + B^6 I^3$$

$$\underline{w}^1(1^\backprime,1) = \frac{\nabla x}{1}(B^3 + B^3) + \frac{\nabla\lambda}{1}(B^3 + B^3) + \frac{\nabla x_5}{4}D^1 + \frac{\nabla\lambda_5}{4}D^3 + \frac{\nabla x \nabla \lambda}{5}D^6$$

$$\overline{M}_2(i,j) = \frac{1}{\Delta x}(B_2 + B_8) + \frac{1}{\Delta y}(B_2 + B_8) + \frac{4}{\Delta x^2}D_2 + \frac{4}{\Delta y^2}D_2 + \frac{2}{\Delta x \Delta y}D_8$$
$$+ B_2 I_1 + B_2 I_2 + B_8 I_3$$
$$+ B^8 I^1 + B^8 I^3 + B^8 I^3$$

$$\underline{w}^\varrho(1^\backprime,1) = \frac{\nabla x}{1}(B^8 + B^8) + \frac{\nabla\lambda}{1}(B^9 + B^9) + \frac{\nabla x_5}{4}D^9 + \frac{\nabla\lambda_5}{4}D^9 + \frac{\nabla x \nabla \lambda}{5}D^9$$

$$\overline{Q}_1(i,j) = A_5\left(\frac{1}{\Delta y}+1\right) + A_5\left(\frac{1}{\Delta x}+1\right)$$

$$\overline{Q}_2(i,j) = A_4\left(\frac{1}{\Delta y}+1\right) + A_5\left(\frac{1}{\Delta x}+1\right)$$

And I_1, I_2 and I_3 are as follows:

$$I_1 = \frac{1}{2\Delta x^2}|w(i+1,j) - w(i-1,j)|$$

$$I_2 = \frac{1}{2\Delta y^2}|w(i,j+1) - w(i,j-1)|$$

$$I_3 = \frac{1}{2\Delta x \Delta y}\left[|w(i+1,j) - w(i-1,j)| + |w(i,j+1) - w(i,j-1)|\right]$$

Conclusion

The DRM program is designed for the analysis of rectangular plates irrespective of material, geometry, edge conditions. The functions of the program are as follows: read the data file; compute the stiffness of the laminate, the fictitious densities, the velocities and displacements and the mid-plane deflections and stresses; check the stability of the numerical computations, the convergence of the solution, and the wrong convergence; compute through-thickness

stresses in direction of plate axes; and transform through-thickness stresses in the lamina principal axes. The convergence of the DRM solution is checked, at the end of each iteration, by comparing the velocities over the plate domain with a predetermined value, which ranges between 10^{-9} for small deflection and 10^{-6} for large deflection. When all velocities are smaller than the predetermined value, the solution is deemed converged and consequently the iterative procedure is terminated. Sometimes DRM solution converges to invalid solution. To check for that the profile of variable is compared with the expected profile over the domain. For example when the value of the function on the boundaries is zero, and it is expected to increase from edge to center, then the solution should follow a similar profile. And when the computed profile is different from that expected, the solution is considered incorrect and can hardly be made to converge to the correct answer by altering the damping coefficients and time increment. Therefore, the boundary conditions should be examined and corrected if they are improper.

Time increment is a very important factor for speeding convergence and controlling numerical computations. When the increment is too small, the convergence becomes tediously slow; and when it is too large, the solution becomes unstable. The proper time increment in the present study it is taken as 0.8 for all boundary conditions. The optimum damping coefficient is that which produces critical motion. When the damping coefficients are large, the motion is over damped and the convergence becomes very slow. And when the coefficients are small, the motion is under damped and can cause numerical instability. Therefore, the damping coefficients must be selected carefully to eliminate under damping and over damping. The errors inherent in the DRM technique include the discretization error that is due to the replacement of a continuous function with a discrete function, and an additional error because the discrete equations are not solved exactly due to the variations of the velocities from the edge of the plate to the center. Finer meshes reduce the discretization error, but increase the round-off error due to the large number of calculations involved. The last type of error is relative to the rank of the interpolating function employed i.e. quadratic, cubic, etc. Refer to Osama Khayal [10-12].

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