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EFFECT OF LAMINATION SCHEME ON BUCKLING LOAD FOR LAMINATED COMPOSITE DECKS PLATES

Osama Mohammed Elmardi Suleiman, Mahmoud Yassin Osman, Tagelsir Hassan

Nile Valley University, Dept. of Mechanical Engineering, Sudan
e-mail: osamamm64@gmail.com

ABSTRACT: Finite element (FE) method is presented for the analysis of thin rectangular laminated composite decks plates under the biaxial action of in – plane compressive loading. The analysis uses the classical laminated plate theory (CLPT) which does not account for shear deformations. In this theory it is assumed that the laminate is in a state of plane stress, the individual lamina is linearly elastic, and there is perfect bonding between layers. The classical laminated plate theory (CLPT), which is an extension of the classical plate theory (CPT) assumes that normal to the mid – surface before deformation remains straight and normal to the mid – surface after deformation. Therefore, this theory is only adequate for buckling analysis of thin laminates. A Fortran program has been developed. New numerical results are generated for in – plane compressive biaxial buckling which serve to quantify the effects of lamination scheme on buckling loading. The results indicate that the symmetric laminate is stiffer than the anti – symmetric one. This phenomenon is caused by coupling between bending and stretching which lowers the buckling loads of symmetric laminate.

KEYWORDS: Lamination scheme, biaxial buckling, classical laminated plate theory, finite element, Fortran program, composite laminated decks plates

1 INTRODUCTION

The objective of this research paper is to present a complete and up to date treatment of uniform cross section rectangular laminated decks plates on buckling. Finite element (FE) method is used for solving governing equations of thin laminated composite plates and their solution using classical laminated plate theory (CLPT). Plates are common structural elements of most engineering structures, including aerospace, automotive, and civil engineering structures, and their study from theoretical and experimental analyses points of view are fundamental to the understanding of the behavior of such structures.

The motivation that led to the carrying out of the present study has come from many years of studying classical laminated plate theory (CLPT) and its analysis by the finite element (FE) method, and also from the fact that there does not exist a publication that contains a detailed coverage of classical laminated plate theory and finite element method in one volume. The present study is an attempt to fulfill the need for a complete treatment of classical laminated theory of plates and its solution by a numerical solution.

The material presented is intended to serve as a basis for a critical study of the fundamentals of elasticity and several branches of solid mechanics including advanced mechanics of materials, theories of plates, composite materials and numerical methods

The problem of critical buckling loads of laminated composite plates is analyzed and solved using the energy method which is formulated by a finite element model. In that model, a four noded rectangular elements of a plate is considered. Each element has three degrees of freedom at each node. The degrees of freedom are the lateral displacement w, and the rotations ϕ and ψ about the y and x axes respectively.

The effects of lamination scheme on the non – dimensional critical buckling loads of laminated composite plates are investigated.
The material chosen has the following properties:

$$E_1/E_2 = 5, 10, 20, 25, 40\ ;\ G_{12} = G_{13} = G_{23} = 0.5E_2\ ;\ \nu_{12} = 0.25\ .$$

Several numerical methods could be used in this study, but the main ones are finite difference method (FDM), dynamic relaxation coupled with finite difference method (DR) as is shown in references [1] – [8], and finite element method (FEM).

In the present work, a numerical method known as finite element method (FEM) is used. It is a numerical procedure for obtaining solutions to many of the problems encountered in engineering analysis. It has two primary subdivisions. The first utilizes discrete elements to obtain the joint displacements and member forces of a structural framework. The second uses the continuum elements to obtain approximate solutions to heat transfer, fluid mechanics, and solid mechanics problem. The formulation using the discrete element is referred to as matrix analysis of structures and yields results identical with the classical analysis of structural frameworks. The second approach is the true finite element method. It yields approximate values of the desired parameters at specific points called nodes. A general finite element computers program, however, is capable of solving both types of problems and the name" finite element method" is often used to denote both the discrete element and the continuum element formulations.

The finite element method combines several mathematical concepts to produce a system of linear and non – linear equations. The number of equations

is usually very large, anywhere from 20 to 20,000 or more and requires the computational power of the digital computer.

It is impossible to document the exact origin of the finite element method because the basic concepts have evolved over a period of 150 or more years. The method as we know it today is an outgrowth of several papers published in the 1950th that extended the matrix analysis of structures to continuum bodies. The space exploration of the 1960th provided money for basic research, which placed the method of a firm mathematical foundation and stimulated the development of multi – purpose computer programs that implemented the method. The design of airplanes, unmanned drones, missiles, space capsules, and the like, provided application areas.

The finite element method (FEM) is a powerful numerical method, which is used as a computational technique for the solution of differential equations that arise in various fields of engineering and applied sciences. The finite element method is based on the concept that one can replace any continuum by an assemblage of simply shaped elements, called finite elements with well-defined force, displacement, and material relationships. While one may not be able to derive a closed – form solution for the continuum, one can derive approximate solutions for the element assemblage that replaces it. The approximate solutions or approximation functions are often constructed using ideas from interpolation theory, and hence they are also called interpolation functions. For more details refer to References [9], [10] and [11].

2 MATHEMATICAL FORMULATIONS

2.1 Introduction

The following assumptions were made in developing the mathematical formulations of laminated plates:

1. All layers behave elastically;
2. Displacements are small compared with the plate thickness;
3. Perfect bonding exists between layers;
4. The laminate is equivalent to a single anisotropic layer;
5. The plate is flat and has a constant thickness;
6. The plate buckles in a vacuum and all kinds of damping are neglected.

Unlike homogeneous plates, where the coordinates are chosen solely based on the plate shape, coordinates for laminated plates should be chosen carefully. There are two main factors for the choice of the coordinate system. The first factor is the shape of the plate. Where rectangular plates will be best represented by the choice of rectangular (i.e. Cartesian) coordinates. It will be relatively easy to represent the boundaries of such plates with coordinates. The second factor is the fiber orientation or orthotropic. If the fibers are set straight within each lamina, then rectangular orthotropic would result. It is possible to set the fibers in a radial and circular fashion, which would result in circular orthotropic.

Indeed, the fibers can also be set in elliptical directions, which would result in elliptical orthotropic.

The choice of the coordinate system is of critical importance for laminated plates. This is because plates with rectangular orthotropic could be set on rectangular, triangular, circular or other boundaries. Composite materials with rectangular orthotropic are the most popular, mainly because of their ease in design and manufacturing. The equations that follow are developed for materials with rectangular orthotropic.

Figure 1 shows the geometry of a plate with rectangular orthotropic drawn in the Cartesian coordinates X, Y, and Z or 1, 2, and 3. The parameters used in such a plate are: (1) the length in the X-direction, (a); (2) the length in the Y – direction (i.e. breadth), (b); and (3) the length in the Z – direction (i.e. thickness), (h).

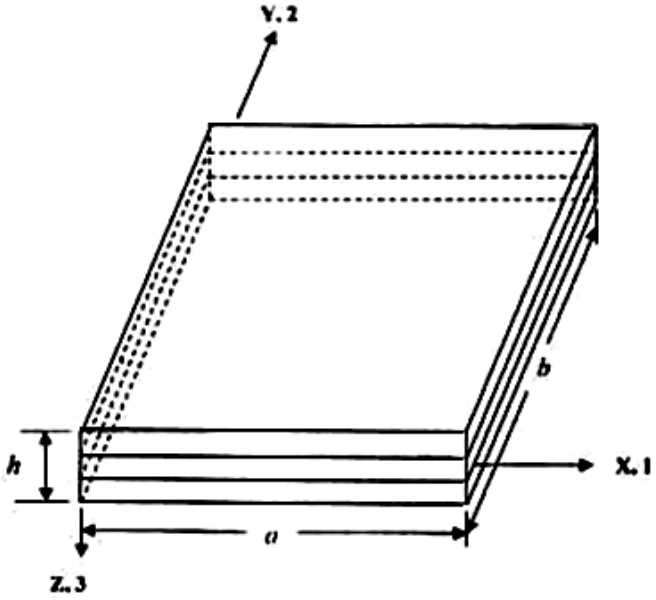


Figure 1. The geometry of a laminated composite plate

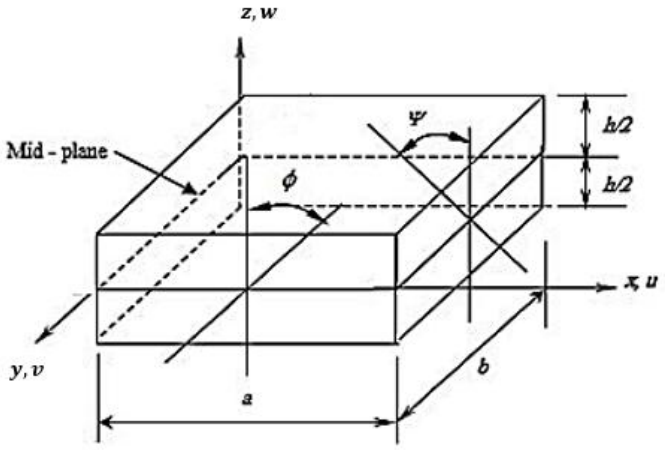
2.2 Fundamental equations of elasticity

Classical laminated plate theory (CLPT) is selected to formulate the problem. Consider a thin plate of length a, breadth b, and thickness h as shown in Figure 2(a), subjected to in – plane loads R_x, R_y and R_{xy} as shown in Figure 2(b). The in – plane displacements $u\ (x, y, z)$ and $v\ (x, y, z)$ can be expressed in terms of the out of plane displacement $w\ (x, y)$ as shown below:

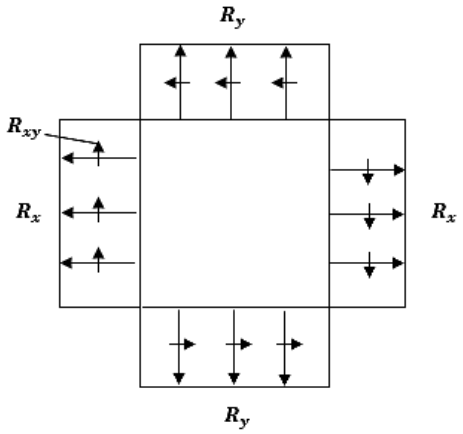
The displacements are:

$$\left.\begin{aligned} u(x,y,z) &= u_o(x,y) - z\frac{\partial w}{\partial x} \\ v(x,y,z) &= v_o(x,y) - z\frac{\partial w}{\partial y} \\ w(x,y,z) &= w_o(x,y) \end{aligned}\right\} \quad (1)$$

Where u_o, v_o and w_o are mid – plane displacements in the direction of the x, y and z axes respectively; z is the perpendicular distance from mid – plane to the layer plane.



(a)



(b)

Figure 2. A plate showing dimensions and deformations

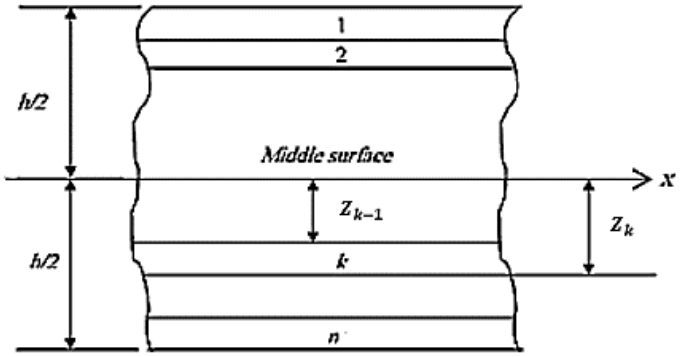


Figure 3. Geometry of an n-layered laminate

The plate shown in Figure 2(a) is constructed of an arbitrary number of orthotropic layers bonded together as in Figure 3.

The strains are:

$$\left.\begin{aligned}\epsilon_x &= \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma &= \frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\end{aligned}\right\} \quad (2)$$

The virtual strains:

$$\left.\begin{aligned}\delta\epsilon_x &= \frac{\partial}{\partial x}\delta u_o - z\frac{\partial^2}{\partial x^2}\delta w + \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta w \\ \delta\epsilon_y &= \frac{\partial}{\partial y}\delta v_o - z\frac{\partial^2}{\partial y^2}\delta w + \frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta w \\ \delta\gamma &= \frac{\partial}{\partial x}\delta v_o + \frac{\partial}{\partial y}\delta u_o - 2z\frac{\partial^2}{\partial x \partial y}\delta w + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial x}\delta w\frac{\partial w}{\partial y}\end{aligned}\right\} \quad (3)$$

The virtual strain energy:

$$\delta U = \int_V \delta\epsilon^T \sigma dV \quad (4)$$

But,

$$\sigma = C\epsilon$$

Where,

$$C = C_{ij}(i, j = 1, 2, 6)$$

$$\therefore\ \delta U = \int_V \delta\epsilon^T\ C\ \delta\epsilon\ dV \tag{5}$$

If we neglect the in-plane displacements u_o and v_o and considering only the linear terms in the strain – displacement equations, we write:

$$\delta\epsilon = -z \begin{vmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x \partial y} \end{vmatrix} \delta w \tag{6}$$

3 THE NUMERICAL METHOD

The finite element is used in this analysis as a numerical method to predict the buckling loads and shape modes of buckling of laminated rectangular plates [12] and [13]. In this method of analysis, four-noded type of elements is chosen. These elements are the four-noded bilinear rectangular elements of a plate. Each element has three degrees of freedom at each node. The degrees of freedom are the lateral displacement (w), and the rotations (ϕ) and (ψ) about the (X) and (Y) axes respectively.

The finite element method is formulated by the energy method. The numerical method can be summarized in the following procedures:

1. The choice of the element and its shape functions.
2. Formulation of finite element model by the energy approach to develop both element stiffness and differential matrices.
3. Employment of the principles of non – dimensionality to convert the element matrices to their non – dimensional forms.
4. Assembly of both element stiffness and differential matrices to obtain the corresponding global matrices.
5. Introduction of boundary conditions as required for the plate edges.
6. Suitable software can be used to solve the problem.

For an n noded element, and 3 degrees of freedom at each node.

Now express w in terms of the shape functions N (given in Appendix (B)) and noded displacements a^e, equation (6) can be written as:

$$\delta\epsilon = -zB\delta a^e \tag{7}$$

Where,

$$B^T = \left[\frac{\partial^2 N_i}{\partial x^2} \quad \frac{\partial^2 N_i}{\partial y^2} \quad 2\frac{\partial^2 N_i}{\partial x \partial y}\right]$$

and

$$N_i a_i^e = [w_i] \qquad i = 1, n$$

The stress-strain relation is:

$$\sigma = C\,\epsilon$$

Where C are the material properties which could be written as follows:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}$$

Where C_{ij} are given in Appendix (A).

$$\delta U = \int_V (B\delta a^e)^T (Cz^2) B a^e dV$$

Where V denotes volume.

$$\delta U = \delta a^{eT} \int_V B^T D B a^e dx\, dy = \delta a^{eT}\, K^e a^e \tag{8}$$

Where $D_{ij} = \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} C_{ij} Z^2\, dZ$ is the bending stiffness, and K^e is the element stiffness matrix which could be written as follows:

$$K^e = \int B^T D B\, dxdy \tag{9}$$

The virtual work done by external forces can be expressed as follows: Refer to Figure 4.
Denoting the nonlinear part of strain by $\delta\epsilon'$

$$\delta W = \iint \delta\epsilon'^T \sigma' dV = \int \delta\epsilon'^T N\, dxdy \tag{10}$$

Where

$$N^T = \begin{bmatrix} N_x\, N_y\, N_{xy} \end{bmatrix} = \begin{bmatrix} \sigma_x\, \sigma_y\, \tau \end{bmatrix} dZ$$

$$\delta\epsilon' = \begin{bmatrix} \delta\epsilon_x \\ \delta\epsilon_y \\ \delta\gamma \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 \\ 0 & \frac{\partial}{\partial y}\delta w \\ \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \tag{11}$$

Hence,

$$\delta W = \iint \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}^T \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 & \frac{\partial}{\partial y}\delta w \\ 0 & \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} dx\, dy \tag{12}$$

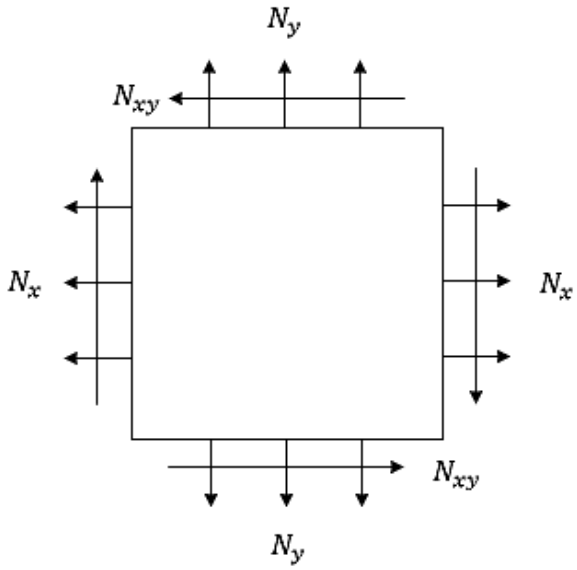


Figure 4. External forces acting on an element

This can be written as:

$$\delta W = \iint \begin{bmatrix} \frac{\partial}{\partial x}\delta w \\ \frac{\partial}{\partial y}\delta w \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} dx\, dy \tag{13}$$

Now $w = N_i a_i^e$

$$\delta W = \delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \tag{14}$$

Substitute $P_x = -N_x, P_y = -N_y, P_{xy} = -N_{xy}$

$$\delta W = -\delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \tag{15}$$

Therefore, equation (15) could be written in the following form:

$$\delta W = -\delta a^{eT} K^D a^e \tag{16}$$

Where,

$$K^D = \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} dx\, dy$$

K^D is the differential stiffness matrix known also as geometric stiffness matrix, initial stress matrix, and initial load matrix.
The total energy:

$$\delta U + \delta W = 0 \tag{17}$$

Since δa^e is an arbitrary displacement which is not zero, then

$$K^e a^e - K^D a^e = 0 \tag{18}$$

Now let us compute the elements stiffness and the differential matrices.

$$K^e = \iint B^T D B\, dx\, dy$$

$$K^e = \iint \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x \partial y} \end{bmatrix}^T \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x \partial y} \end{bmatrix} dx\, dy$$

The elements stiffness matrix can be expressed as follows:

$$\begin{aligned} K_{ij}^e = \iint \Bigg[& D_{11} \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial x^2} + D_{12} \left(\frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial y^2} \right) \\ & + 2D_{16} \left(\frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial x \partial y} \right) + D_{22} \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial y^2} \\ & + 2D_{26} \Bigg(\frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial y^2} \\ & + \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial x \partial y} \Bigg) + 4D_{66} \frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial x \partial y} \Bigg] dx\, dy \end{aligned} \tag{19}$$

The elements differential stiffness matrix can be expressed as follows;

$$\begin{aligned} K_{ij}^D = \iint \Bigg[& P_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \\ & + P_{xy} \left(\frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} \right) + P_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \Bigg] dx dy \end{aligned} \tag{20}$$

The integrals in equations (19) and (20) are given in Appendix (C). The shape local co-ordinate for a 4-noded element is shown below in Figure 5.

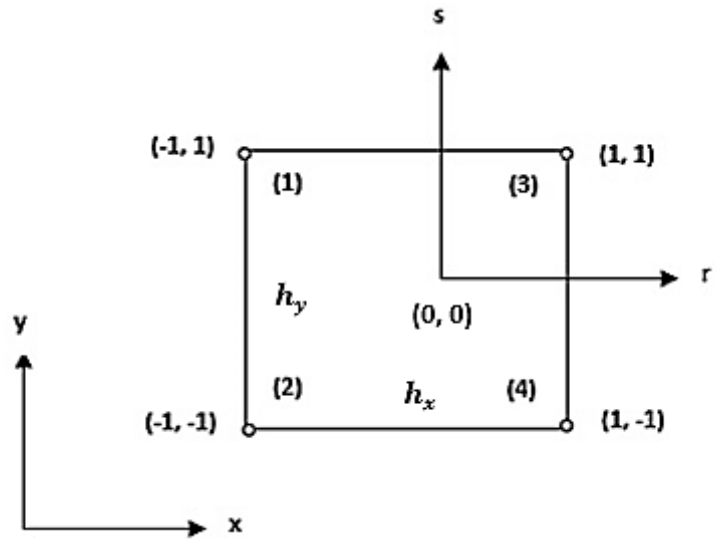


Figure 5. A four noded element with local and global co–ordinates

The shape functions for the 4-noded element expressed in global co-ordinates (x, y) are as follows:

$$w = N_1 w_1 + N_2 \phi_1 + N_3 \psi_1 + N_4 w_2 + N_5 \phi_2 + N_6 \psi_2$$
$$+N_7 w_3 + N_8 \phi_3 + N_9 \psi_3 + N_{10} w_4 + N_{11} \phi_4 + N_{12} \psi_4$$

Where,

$$\phi = \frac{\partial w}{\partial x}\,, \qquad \psi = \frac{\partial w}{\partial y}$$

The shape functions in local co – ordinates are as follows:

$$N_i = a_{i1} + a_{i2} r + a_{i3} s + a_{i4} r^2 + a_{i5} rs + a_{i6} s^2 + a_{i7} r^3 + a_{i8} r^2 s + a_{i9} rs^2$$
$$+a_{i10} s^3 + a_{i11} r^3 s + a_{i12} rs^3$$
$$N_j = a_{j1} + a_{j2} r + a_{j3} s + a_{j4} r^2 + a_{j5} rs + a_{j6} s^2 + a_{j7} r^3 + a_{j8} r^2 s + a_{j9} rs^2$$
$$+a_{j10} s^3 + a_{j11} r^3 s + a_{j12} rs^3$$

The integrals of the shape functions in local co – ordinates are as follows:

$$q_1 = \iint \frac{\partial^2 N_i}{\partial r^2} \frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i4} a_{j4} + 3a_{i7} a_{j7} + \frac{1}{3} a_{i8} a_{j8} + a_{i11} a_{j11}\right]$$

$$q_2 = \iint \frac{\partial^2 N_i}{\partial s^2} \frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i6} a_{j6} + \frac{1}{3} a_{i9} a_{j9} + 3a_{i10} a_{j10} + a_{i12} a_{j12}\right]$$

$$q_3 = \iint \frac{\partial^2 N_i}{\partial r^2} \frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i4} a_{j6} + a_{i7} a_{j9} + a_{i8} a_{j10} + a_{i11} a_{j12}\right]$$

$$q_4 = \iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i6}a_{j4} + a_{i9}a_{j7} + a_{i10}a_{j8} + a_{i12}a_{j11}\right]$$

$$q_5 = \iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r\partial s}\, dr\, ds = 8\Big[a_{i4}a_{j5} + a_{i4}a_{j11} + 2a_{i7}a_{j8} + a_{i4}a_{j12} + \frac{2}{3}a_{i8}a_{j9}\Big]$$

$$q_6 = \iint \frac{\partial^2 N_i}{\partial r\partial s}\frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 8\left[a_{i5}a_{j4} + 2a_{i8}a_{j7} + a_{i11}a_{j4} + \frac{2}{3}a_{i9}a_{j8} + a_{i12}a_{j4}\right]$$

$$q_7 = \iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r\partial s}\, dr\, ds = 8\left[a_{i6}a_{j5} + a_{i6}a_{j11} + \frac{2}{3}a_{i9}a_{j8}\right]$$

$$q_8 = \iint \frac{\partial^2 N_i}{\partial r\partial s}\frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 8\left[a_{i5}a_{j6} + \frac{2}{3}a_{i8}a_{j9} + a_{i11}a_{j6}\right]$$

$$q_9 = \iint \frac{\partial^2 N_i}{\partial r\partial s}\frac{\partial^2 N_j}{\partial r\partial s}\, dr\, ds = 4\left[a_{i5}a_{j5} + a_{i5}a_{j11} + \frac{4}{3}a_{i8}a_{j8} + a_{i5}a_{j12} + \frac{4}{3}a_{i9}a_{j9} + a_{i11}a_{j12} + a_{i12}a_{j11} + \frac{9}{5}a_{i12}a_{j12}\right]$$

$$q_{10} = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\, dr\, ds = 4\Big[a_{i2}a_{j2} + \frac{1}{3}(3a_{i2}a_{j7} + 4a_{i4}a_{j4} + 3a_{i7}a_{j2} + a_{i7}a_{j9} + a_{i5}a_{j5} + a_{i9}a_{j2} + a_{i5}a_{j11} + a_{i7}a_{j9} + \frac{4}{3}a_{i8}a_{j8} + a_{i9}a_{j7} a_{i11}a_{j5}) + \frac{1}{5}(a_{i5}a_{j12} + a_{i9}a_{j9} + a_{i12}a_{j5} + 9a_{i7}a_{j7} + 3a_{i11}a_{j11} + a_{i11}a_{j12} + a_{i12}a_{j11}) + \frac{1}{7}a_{i12}a_{j12}\Big]$$

$$q_{11} = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\, dr\, ds = 4\Big[a_{i3}a_{j3} + \frac{1}{3}(a_{i3}a_{j8} + a_{i5}a_{j5} + a_{i8}a_{j3} + 3a_{i3}a_{j10} + 4a_{i6}a_{j6} + 3a_{i10}a_{j3} + a_{i5}a_{j12} + a_{i8}a_{j10} + \frac{4}{3}a_{i9}a_{j9} + a_{i10}a_{j8} + a_{i12}a_{j5}) + \frac{1}{5}(a_{i5}a_{j11} + a_{i8}a_{j8} + a_{i11}a_{j5} + 9a_{i10}a_{j10} + a_{i11}a_{j12} + a_{i12}a_{j11} + 3a_{i2}a_{j12}) + \frac{1}{7}a_{i11}a_{j11}\Big]$$

$$q_{12} = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial s}\, dr\, ds = 4\Big[a_{i2}a_{j3} + \frac{1}{3}(a_{i2}a_{j8} + 2a_{i4}a_{j5} + 3a_{i7}a_{j8}$$
$$+3a_{i2}a_{j10} + 2a_{i5}a_{j6} + a_{i9}a_{j3} + 2a_{i4}a_{j12} + 3a_{i7}a_{j10} + \frac{4}{3}a_{i8}a_{j9} + \frac{1}{3}a_{i9}a_{j8}$$
$$+2a_{i11}a_{j6})\Big]$$

$$q_{13} = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\, dr\, ds = 4\Big[a_{i3}a_{j2} + \frac{1}{3}(3a_{i3}a_{j7} + 2a_{i5}a_{j4} + a_{i8}a_{j2}$$
$$+a_{i3}a_{j9} + 2a_{i6}a_{j5} + 3a_{i10}a_{j2} + 2a_{i6}a_{j11} + \frac{1}{3}a_{i8}a_{j9} + \frac{4}{3}a_{i9}a_{j8} + 3a_{i10}a_{j7}$$
$$+2a_{i12}a_{j4}) + \frac{1}{5}(2a_{i6}a_{j12} + 3a_{i10}a_{j9} + 3a_{i8}a_{j7} + 2a_{i11}a_{j4})\Big]$$

The values of the integrals are converted from local co-ordinate (r, s) to global co-ordinates.

The integrals of the shape functions in global co-ordinates are as follows:

$$r_1 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial N_j}{\partial x^2}\, dx\, dy = \left(\frac{4h_y}{h_x^3}\right)q_1 = \frac{4n^3 b}{ma^3}q_1$$

$$r_2 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \left(\frac{4h_x}{h_y^3}\right)q_2 = \frac{4am^3}{nb^3}q_2$$

$$r_3 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \left(\frac{4}{h_y h_x}\right)q_3 = \frac{4mn}{ab}q_3$$

$$r_4 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \left(\frac{4}{h_y h_x}\right)q_4 = \frac{4mn}{ab}q_4$$

$$r_5 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \left(\frac{4}{h_x^2}\right)q_5 = \frac{4n^2}{a^2}q_5$$

$$r_6 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \left(\frac{4}{h_x^2}\right)q_6 = \frac{4n^2}{a^2}q_6$$

$$r_7 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \left(\frac{4}{h_y^2}\right)q_7 = \frac{4m^2}{a^2}q_7$$

$$r_8 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \left(\frac{4}{h_y^2}\right)q_8 = \frac{4m^2}{b^2}q_8$$

$$r_9 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \left(\frac{4}{h_y h_x}\right)q_9 = \frac{4mn}{ab}q_9$$

$$r_{10} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx\, dy = \left(\frac{h_y}{h_x}\right) q_{10} = \frac{bn}{am} q_{10}$$

$$r_{11} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\, dx\, dy = \left(\frac{h_x}{h_y}\right) q_{11} = \frac{am}{bn} q_{11}$$

$$r_{12} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\, dx\, dy = q_{12}$$

$$r_{13} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\, dx\, dy = q_{13}$$

In the previous equations $h_x = \frac{a}{n}$ and $h_y = \frac{b}{m}$ where a and b are the lengths of the plate along the x – and y – axis respectively. n and m are the number of elements in the x – and y – directions respectively.

The elements of the stiffness matrix and the differential matrix can be written as follows:

$$K_{ij} = D_{11}r_1 + D_{12}r_4 + 2D_{16}r_3 + D_{12}r_3 + D_{22}r_2 + 2D_{66}r_8 + 2D_{16}r_5 + 2D_{26}r_7 + 4D_{66}r_9$$

$$K_{ij}^D = P_x r_{10} + P_{xy}(r_{12} + r_{13}) + P_y r_{11}$$

or in the non – dimensional form:

$$K_{ij} = \frac{4n^3}{m}\left(\frac{b}{a}\right)\overline{D}_{11}q_1 + 4mn\left(\frac{a}{b}\right)\overline{D}_{12}q_4 + 4n^2\overline{D}_{16}q_6 + 4mn\left(\frac{a}{b}\right)\overline{D}_{12}q_3$$
$$+\frac{4m^3}{n}\left(\frac{a}{b}\right)\overline{D}_{22}q_2 + 4m^2\left(\frac{a}{b}\right)^2\overline{D}_{26}q_8 + 4n^2\overline{D}_{16}q_5 + 4m^2\left(\frac{a}{b}\right)^2\overline{D}_{26}q_7$$
$$+4mn\left(\frac{a}{b}\right)\overline{D}_{66}q_9$$

$$K_{ij}^D = \bar{P}_x\frac{n}{m}\left(\frac{b}{a}\right)q_{10} + \bar{P}_{xy}(q_{12} + q_{13}) + \bar{P}_y\frac{m}{n}\left(\frac{a}{b}\right)q_{11}$$

where

$$\overline{D}_{ij} = \left(\frac{1}{E_1 h^3}\right)D_{ij}\,, \quad \bar{P}_i = \left(\frac{a}{E_1 h^3}\right)P_i$$

Also

$$\bar{w} = \left(\frac{1}{h}\right)w, \qquad \bar{\phi} = \left(\frac{h}{a}\right)\phi, \qquad \bar{\psi} = \left(\frac{h}{a}\right)\psi, \qquad \bar{b} = b/a$$

The transformed stiffnesses are as follows:

$$C_{11} = C'_{11}c^4 + 2c^2s^2(C'_{11} + 2C'_{66}) + C'_{22}s^4$$

$$C_{12} = c^2 s^2 (C'_{11} + C'_{22} + 4C'_{66}) + C'_{12}(c^4 + s^4)$$
$$C_{16} = cs[C'_{11} c^4 + C'_{22} s^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{22} = C'_{11} s^4 + 2c^2 s^2 (C'_{12} + 2C'_{66}) + C'_{22} c^4$$
$$C_{26} = cs[C'_{11} s^2 - C'_{22} c^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{66} = (C'_{11} + C'_{22} - 2C'_{12}) c^2 s^2 + C'_{66}(c^2 - s^2)^2$$

Where

$$C'_{11} = \frac{E_1}{1 - v_{12} v_{21}}$$
$$C'_{12} = \frac{v_{21}\, E_1}{1 - v_{12} v_{21}} = \frac{v_{12}\, E_1}{1 - v_{12} v_{21}}$$
$$C'_{22} = \frac{E_2}{1 - v_{12} v_{21}}$$
$$C'_{44} = G_{23}\,, \qquad C'_{55} = G_{13} \quad \text{and} \quad C'_{66} = G_{12}$$

E_1 and E_2 are the elastic moduli in the direction of the fiber and the transverse directions respectively, v is the Poisson's ratio. G_{12}, G_{13}, and G_{23} are the shear moduli in the $x - y$ plane, $y - z$ plane, and $x - z$ plane respectively, and the subscripts 1 and 2 refer to the direction of fiber and the transverse direction respectively.

4 EFFECT OF LAMINATION SCHEME

In the present analysis the lamination scheme of plates is supposed to be symmetric, anti – symmetric and quasi – isotropic.

Four lamination schemes were considered which are symmetric and anti – symmetric cross – ply and angle – ply laminates. Table 1 gives a comparison between the non – dimensional buckling loads for all lamination schemes. The results are shown graphically in Figure 6. The thickness of all layers is assumed equal, the length to thickness ratio ($a/h = 20$), and the modulus ratio ($E_1/E_2 = 5$). It is noticed from Table 1 and Figures 6, 7 and 8 that the values of the non – dimensional buckling loads for both symmetric and anti – symmetric lamination are slightly different, except for symmetric and anti – symmetric angle – ply laminates which are exactly the same. Because of this fact, the rest of the upcoming effects will be discussed for symmetric case only. The results indicate that the symmetric laminate is stiffer than the anti – symmetric one. This phenomenon is caused by coupling between bending and stretching which lowers the buckling loads of symmetric laminate.

Table 1. The first five non-dimensional buckling loads $\bar{P} = Pa^2/E_1h^3$ of symmetric cross-ply (0/ 90/ 90/ 0) and anti-symmetric cross – ply (0/ 90/ 0/ 90), and symmetric angle-ply (45/ -45/ -45/ 45) and anti-symmetric angle-ply (45/ -45/ 45/ -45) laminated plates with $a/h = 20$, and $E_1/E_2 = 5$

Lamination Scheme	Mode Number	Boundary Conditions		
		SS	CC	CS
0/ 90/ 90/ 0	1	0.6972	2.1994	1.8225
	2	1.2522	2.5842	2.0097
	3	2.4284	4.1609	2.7116
	4	2.6907	4.7431	4.3034
	5	2.7346	5.0168	4.4536
0/ 90/ 0/ 90	1	0.6973	2.2273	1.5591
	2	1.9947	3.9687	2.3391
	3	1.9958	3.9732	3.7581
	4	2.6912	4.7871	3.8290
	5	4.3962	7.0544	4.5402
45/-45/-45/45	1	0.8729	1.9505	1.4756
	2	1.6400	2.8534	2.1162
	3	2.3130	3.8941	3.3039
	4	2.7100	4.3753	3.3068
	5	3.5488	5.2694	4.4166
45/-45/45/-45	1	0.8729	2.2010	1.6554
	2	1.6400	3.7616	2.5672
	3	2.3130	3.7654	3.4642
	4	2.7100	5.6599	4.2174
	5	3.5488	5.9540	4.8091

Tables 2 and 3 show the buckling load of quasi – isotropic rectangular composite plate with $a/h = 20$, $a/b = 1$ and different modulus ratios ($E_1/E_2 = 40$ and 5). The buckling load is highly influenced by its boundary conditions. The buckling load of the quasi – isotropic (0/+45/-45/90) rectangular composite plate with CC type boundary condition is 1.5 times higher than the buckling load of the composite plate with CS type boundary condition and more than 3 times of SS type boundary condition.

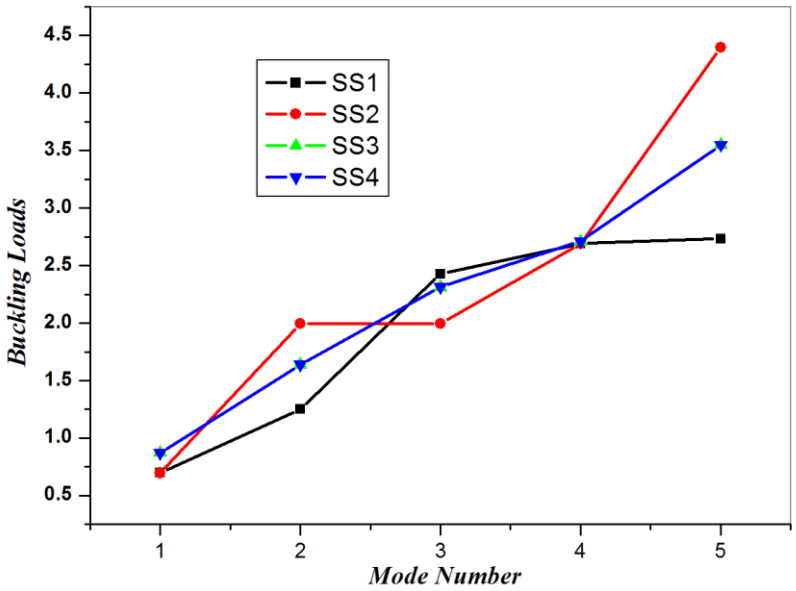


Figure 6. Effect of lamination scheme for simply supported laminates

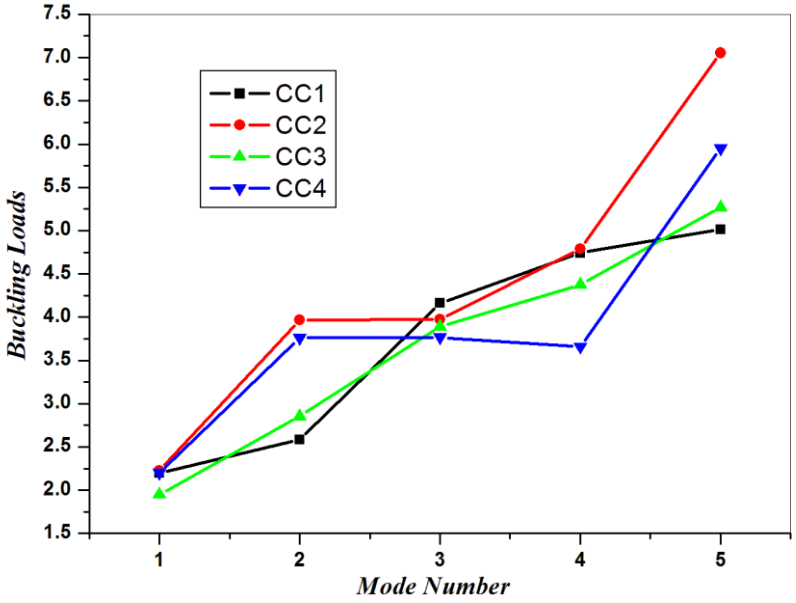


Figure 7. Effect of lamination scheme for clamped – clamped laminates

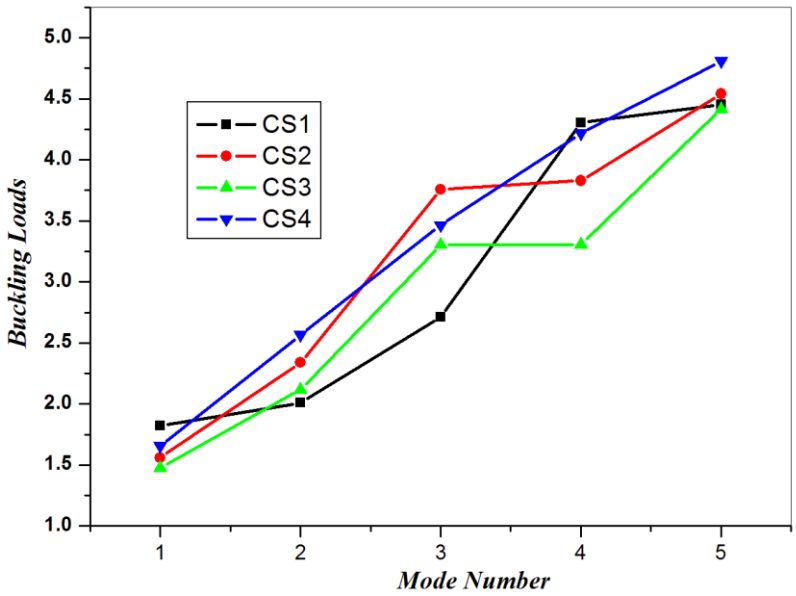


Figure 8. Effect of lamination scheme for clamped – simply supported laminates

Table 2. The first three non-dimensional buckling loads of quasi-isotropic (0/+45/-45/90) laminated plates with a/h=20, and $E_1/E_2 = 40$

Mode Number	Boundary Conditions		
	SS	CC	CS
1	0.4905	1.6878	1.1683
2	1.4842	3.0187	1.7359
3	1.4850	3.0229	2.7673

Table 3. The first three non-dimensional buckling load of quasi-isotropic (0/+45/-45/90) laminated plates with a/h=20, and $E_1/E_2 = 5$

Mode Number	Boundary Conditions		
	SS	CC	CS
1	0.7338	2.2255	1.5717
2	2.0202	3.9506	2.3714
3	2.0214	3.9549	3.7214

5 CONCLUDING REMARKS

A Fortran program based on finite elements (FE) has been developed for buckling analysis of thin rectangular laminated decks plates using classical laminated plate theory (CLPT). The problem of buckling loads of generally layered composite plates has been studied. The problem is analyzed and solved using the energy approach, which is formulated by a finite element model. In

this method, quadrilateral elements are applied utilizing a four noded model. Each element has three degrees of freedom at each node. The degrees of freedom are: lateral displacement (w), and rotation (ϕ) and (ψ) about the x and y axes respectively. The finite element model has been formulated to compute the buckling loads of laminated plates with rectangular cross – section and to study the effects of lamination scheme on the non – dimensional critical buckling loads of laminated composite plates. New results have been presented. These results show that the symmetric laminate is stiffer than the anti – symmetric one. This phenomenon is caused by coupling between bending and stretching which lowers the buckling loads of symmetric laminate.

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